# CHAPTER 11
# QUEUEING MODELS
# SOLUTION TO SOLVED PROBLEMS

## 11.S1 Managing Waiting Lines at First Bank of Seattle

*Sally Gordon has just completed her MBA degree and is proud to have earned a promotion to Vice President for Customer Services at the First Bank of Seattle. One of her responsibilities is to manage how tellers provide services to customers, so she is taking a hard look at this area of the bank's operations. Customers needing teller service arrive randomly at a mean rate of 30 per hour. Customers wait in a single line and are served by the next available teller when they reach the front of the line. Each service takes a variable amount of time (assume an exponential distribution), but on average can be completed in 3 minutes. The tellers earn an average wage of $18 per hour.*

a. *If two tellers are used, what will be the average waiting time for a customer before reaching a teller? On average, how many customers will be in the bank, including those currently being served?*

This is an M/M/s queueing system. The mean arrival rate is $\lambda = 30$ customers per hour. The mean service rate is $\mu = (60 \text{ min/hr}) / (3 \text{ minutes/customer}) = 20$ customers per hour. The number of servers is $s = 2$. The Template for the M/M/s queueing model is shown below. The average waiting time for a customer before reaching a teller is $W_q = 0.064$ hours $= 3.86$ minutes. The average number of customers in the bank, including those currently being served, is $L = 3.43$.

| | A | B | C | D | E | G | H | I |
|---|---|---|---|---|---|---|---|---|
| 1 | **Template for the M/M/s Queueing Model** | | | | | | | |
| 2 | | | | | | | | |
| 3 | | | **Data** | | | | **Results** | |
| 4 | | $\lambda$ = | 30 | (mean arrival rate) | | L = | 3.428571429 | |
| 5 | | $\mu$ = | 20 | (mean service rate) | | $L_q$ = | 1.928571429 | |
| 6 | | s = | 2 | (# servers) | | | | minutes |
| 7 | | | | | | W = | 0.114285714 | 6.857142857 |
| 8 | | Pr(W > t) = | 0.50480467 | | | $W_q$ = | 0.064285714 | 3.857142857 |
| 9 | | when t = | 0.08333333 | | | | | |
| 10 | | | | | | $\rho$ = | 0.75 | |
| 11 | | Prob($W_q$ > t) = | 0.27938456 | | | | | |
| 12 | | when t = | 0.08333333 | | | n | $P_n$ | |
| 13 | | | | | | 0 | 0.142857143 | |

b. *Company policy is to have no more than a 10% chance that a customer will need to wait more than 5 minutes before reaching a teller. How many tellers need to be used in order to meet this standard?*

As in part *a*, this is an M/M/s queueing system with mean arrival rate $\lambda = 30$ customers per hour and mean service rate $\mu = 20$ customers per hour. To satisfy the company policy that there be no more than a 10% chance that a customer will need to wait more than 5 minutes before reaching a teller, we need to assure that $\Pr(W_q > t)$ in cell C11 is no more than 10% when $t = 0.0833$ hours (5 minutes). From part *a*, when $s = 2$, $\Pr(W_q > t) = 27.9\%$. As shown below, when $s = 3$, $\Pr(W_q > t) = 1.9\%$. Thus, First Bank will need at least 3 tellers to meet this standard.

| | A | B | C | D | E | G | H |
|---|---|---|---|---|---|---|---|
| 1 | **Template for the M/M/s Queueing Model** | | | | | | |
| 2 | | | | | | | |
| 3 | | | **Data** | | | | **Results** |
| 4 | | $\lambda$ = | 30 | (mean arrival rate) | | L = | 1.736842105 |
| 5 | | $\mu$ = | 20 | (mean service rate) | | $L_q$ = | 0.236842105 |
| 6 | | s = | 3 | (# servers) | | | |
| 7 | | | | | | W = | 0.057894737 |
| 8 | | Pr(W > t) = | 0.23946063 | | | $W_q$ = | 0.007894737 |
| 9 | | when t = | 0.08333333 | | | | |
| 10 | | | | | | $\rho$ = | 0.5 |
| 11 | | Prob($W_q$ > t) = | 0.01944118 | | | | |
| 12 | | when t = | 0.08333333 | | | n | $P_n$ |
| 13 | | | | | | 0 | 0.210526316 |

c. *Sally feels that a significant cost is incurred by making a customer wait because of potential lost future business. Sally estimates the cost to be \$0.50 for each minute a customer spends in the bank, counting both waiting time and service time. Given this cost, how many tellers should Sally employ?*

This question can be answered using the template for economic analysis of the M/M/s queueing model. The cost of service is $C_s$ = \$18/hour/server. The cost of waiting is $C_w$ = \$0.50/minute/server = \$30/hour/server.

The total cost of service and waiting is $C_s\, s + C_w\, L$. As seen in the templates below, with 2 servers the total cost is \$138.86. With 3 servers, the total cost is \$106.11. With 4 servers the total cost is \$118.34. Therefore, Sally should employ 3 servers.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Template for Economic Analysis of M/M/s Queueing Model** | | | | | | |
| 2 | | | | | | | |
| 3 | | | **Data** | | | | **Results** |
| 4 | | $\lambda$ = | 30 | (mean arrival rate) | | L = | 3.428571429 |
| 5 | | $\mu$ = | 20 | (mean service rate) | | $L_q$ = | 1.928571429 |
| 6 | | s = | 2 | (# servers) | | | |
| 7 | | | | | | W = | 0.114285714 |
| 8 | | Pr(W > t) = | 0.50480467 | | | $W_q$ = | 0.064285714 |
| 9 | | when t = | 0.08333333 | | | | |
| 10 | | | | | | $\rho$ = | 0.75 |
| 11 | | Prob($W_q$ > t) = | 0.27938456 | | | | |
| 12 | | when t = | 0.08333333 | | | n | $P_n$ |
| 13 | | | | | | 0 | 0.142857143 |
| 14 | | **Economic Analysis:** | | | | 1 | 0.214285714 |
| 15 | | Cs = | $18.00 | (cost / server / unit time) | | 2 | 0.160714286 |
| 16 | | Cw = | $30.00 | (waiting cost / unit time) | | 3 | 0.120535714 |
| 17 | | | | | | 4 | 0.090401786 |
| 18 | | Cost of Service | $36.00 | | | 5 | 0.067801339 |
| 19 | | Cost of Waiting | $102.86 | | | 6 | 0.050851004 |
| 20 | | Total Cost | $138.86 | | | 7 | 0.038138253 |

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Template for Economic Analysis of M/M/s Queueing Model** | | | | | | |
| 2 | | | | | | | |
| 3 | | | **Data** | | | | **Results** |
| 4 | | $\lambda$ = | 30 | (mean arrival rate) | | L = | 1.736842105 |
| 5 | | $\mu$ = | 20 | (mean service rate) | | $L_q$ = | 0.236842105 |
| 6 | | s = | 3 | (# servers) | | | |
| 7 | | | | | | W = | 0.057894737 |
| 8 | | Pr(W > t) = | 0.23946063 | | | $W_q$ = | 0.007894737 |
| 9 | | when t = | 0.08333333 | | | | |
| 10 | | | | | | $\rho$ = | 0.5 |
| 11 | | Prob($W_q$ > t) = | 0.01944118 | | | | |
| 12 | | when t = | 0.08333333 | | | n | $P_n$ |
| 13 | | | | | | 0 | 0.210526316 |
| 14 | | **Economic Analysis:** | | | | 1 | 0.315789474 |
| 15 | | Cs = | $18.00 | (cost / server / unit time) | | 2 | 0.236842105 |
| 16 | | Cw = | $30.00 | (waiting cost / unit time) | | 3 | 0.118421053 |
| 17 | | | | | | 4 | 0.059210526 |
| 18 | | Cost of Service | $54.00 | | | 5 | 0.029605263 |
| 19 | | Cost of Waiting | $52.11 | | | 6 | 0.014802632 |
| 20 | | Total Cost | $106.11 | | | 7 | 0.007401316 |

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Template for Economic Analysis of M/M/s Queueing Model** | | | | | | |
| 2 | | | | | | | |
| 3 | | | **Data** | | | | **Results** |
| 4 | | $\lambda$ = | 30 | (mean arrival rate) | | L = | 1.544751381 |
| 5 | | $\mu$ = | 20 | (mean service rate) | | $L_q$ = | 0.044751381 |
| 6 | | s = | 4 | (# servers) | | | |
| 7 | | | | | | W = | 0.051491713 |
| 8 | | Pr(W > t) = | 0.1974963 | | | $W_q$ = | 0.001491713 |
| 9 | | when t = | 0.08333333 | | | | |
| 10 | | | | | | $\rho$ = | 0.375 |
| 11 | | Prob($W_q$ > t) = | 0.00115636 | | | | |
| 12 | | when t = | 0.08333333 | | | n | $P_n$ |
| 13 | | | | | | 0 | 0.220994475 |
| 14 | | **Economic Analysis:** | | | | 1 | 0.331491713 |
| 15 | | Cs = | $18.00 | (cost / server / unit time) | | 2 | 0.248618785 |
| 16 | | Cw = | $30.00 | (waiting cost / unit time) | | 3 | 0.124309392 |
| 17 | | | | | | 4 | 0.046616022 |
| 18 | | Cost of Service | $72.00 | | | 5 | 0.017481008 |
| 19 | | Cost of Waiting | $46.34 | | | 6 | 0.006555378 |
| 20 | | Total Cost | $118.34 | | | 7 | 0.002458267 |

d. *First Bank has two types of customers: merchant customers and regular customers. The mean arrival rate for each type of customer is 15 per hour. Both types of customers currently wait in the same line and are served by the same tellers with the same average service time. However, Sally is considering changing this. The new system she is considering would have two lines—one for merchant customers and one for regular customers. There would be a single teller serving each line. What would be the average waiting time for each type of customer before reaching a teller? On average, how many total customers would be in the bank, including those currently being served? How do these results compare to those from part* a.

The queueing systems for merchant and regular customers are two separate, but identical, M/M/s queueing systems. For each, the mean arrival rate is $\lambda = 15$ customers per hour. The mean service rate is $\mu = 20$ customers per hour. The number of servers is $s = 1$. The template for the M/M/s queueing model is shown below. The average waiting time for a customer before reaching a teller is $W_q = 0.15$ hours = 9 minutes. The average number of customers of each type in the bank, including those currently being served, is $L = 3$. Thus, the total number of customers (of both types) is 6. These results are significantly worse than those from part *a*.

e. *Sally feels that if the tellers are specialized into merchant tellers and regular tellers, they would be more efficient and could serve customers in an average of 2.5 minutes instead of 3 minutes. Answer the questions for part* d *again with this new average service time.*

Like part *d*, the queueing systems for merchant and regular customers are two separate, but identical, M/M/s queueing systems. For each, the mean arrival rate is $\lambda$ = 15 customers per hour. The mean service rate is $\mu$ = (60 min/hr) / (2.5 minutes/customer) = 24 customers per hour. The number of servers is $s$ = 1. The template for the M/M/s queueing model is shown below. The average waiting time for a customer before reaching a teller is $W_q$ = 0.069 hours = 4.17 minutes. The average number of customers of each type in the bank, including those currently being served, is $L$ = 1.67. Thus, the total number of customers (of both types) is 3.33. These results are similar to those from part *a*. The waiting time before reaching a teller is slightly higher (4.17 minutes vs. 3.86 minutes), but the total number of customers in the bank is slightly smaller (3.33 vs. 3.43).